

ON A PROOF OF THE BOUCHARD-SULKOWSKI CONJECTURE

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ABSTRACT. In this short note, we give a proof of the free energy part of the BKMP conjecture of \mathbb{C}^3 proposed by Bouchard and Sulkowski [4]. Hence the proof of the full BKMP conjecture for the case of \mathbb{C}^3 has been finished.

1. INTRODUCTION

Motivated by B. Eynard and his collaborators' series works on matrix model [9, 7, 8], V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti [1] proposed a new approach (Remodeling the B-model) to compute the topological string amplitudes for local Calabi-Yau manifolds and conjectured that the remodeling approach is equivalent to the Gromov-Witten theory of corresponding toric Calabi-Yau manifolds [2]. In particular, for the case of \mathbb{C}^3 , V. Bouchard and M. Mariño [3] calculated the correlation functions by remodeling approach and conjectured that they are equal to the topological vertex computed by Gromov-Witten theory. Later, L. Chen and J. Zhou [6, 15] gave the rigorous proof independently based on the symmetric form of cut-and-join equation of Mariño-Vafa formula proved in [12] (see also [10] for a new proof).

Recently, V. Bouchard and P. Sulkowski [4] proposed the following free energy part of the BKMP conjecture for the case of \mathbb{C}^3 (Conjecture 2 in [4]).

Conjecture 1.1. *Let Σ_f be the framed curve mirror to $X = \mathbb{C}^3$. Then the free energies obtained through the Eynard-Orantin recursion are given by:*

$$F^{(g)} = \frac{1}{2}(-1)^g \times \frac{|B_{2g}||B_{2g-2}|}{2g(2g-2)(2g-2)!}.$$

In this note, we will give a proof of the Conjecture 1.1 based on a Hodge integral identity and some residue calculations. After finished the draft of this paper and contacted with Prof. V. Bouchard, the author knew that this conjecture has also been proved by V. Bouchard and his collaborators [5] at the same time.

2. THE BKMP CONJECTURE

Let us consider a Riemann surface with genus \bar{g} ,

$$\Sigma = \{x, y \in \mathbb{C}^* | H(x, y; z_a) = 0\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

where $z_a, a = 1, \dots, k$ are the deformation parameters of the complex structure of Σ . $H(x, y; z_a)$ is a polynomial in (x, y) which are \mathbb{C}^* -variables. Let $q_i, i = 1, \dots, 2\bar{g} + 2$ be the ramification points of Σ and on the neighborhood of q_i , one can find two distinct points $q, \bar{q} \in \Sigma$ such that $x(q) = x(\bar{q})$. We mention that the mirror curve of a toric Calabi-Yau 3-fold satisfies these conditions.

The recursion process starts with the following ingredients.

2.1. Ingredients. First, one needs the meromorphic differential $\omega(p) = \log y(p) \frac{dx(p)}{x(p)}$ on Σ .

One also needs the Bergmann kernel $B(p, q)$ on Σ defined by the following conditions.

$$B(p, q) \sim_{p \rightarrow q} \frac{dpdq}{(p - q)^2} + \text{finite.}$$

$B(p, q)$ is holomorphic except $p = q$.

$$\oint_{A_\alpha} B(p, q) = 0, \quad \alpha = 1, \dots, \bar{g}.$$

where (A_α, B^α) is a canonical symplectic basis of one-cycles on Σ .

For the case of $\bar{g} = 0$, the Bergman kernel is given by

$$B(p_1, p_2) = \frac{dy_1 dy_2}{(y_1 - y_2)^2}, \quad y_i = y(x_i).$$

2.2. BKMP's construction. Inspired by the work [14], Bouchard, Klemm, Mariño and Pasquetti [1, 2] defined the free energy $F^{(g,h)}(p_1, \dots, p_h)$ on the mirror curve $H(x, y; z_a) = 0$ based on the topological recursions constructed by B. Eynard and N. Orantin [7] as follows.

$$F^{(g,h)}(p_1, \dots, p_h) = \int W^{(g,h)}(p_1, \dots, p_h),$$

$$W^{(0,1)}(p) = \omega(p),$$

$$W^{(0,2)}(p_1, p_2) = B(p_1, p_2) - \frac{dp_1 dp_2}{(p_1 - p_2)^2},$$

$$W^{(g,h)}(p_1, \dots, p_h) = \tilde{W}^{g,h}(p_1, \dots, p_h), \quad \text{for } (g, h) \neq (0, 1), (0, 2).$$

where $\tilde{W}^{(g,h)}(p_1, \dots, p_h)$ is a multilinear meromorphic differential defined by the following topological recursions.

$$\tilde{W}^{(0,1)}(p) = 0, \quad \tilde{W}^{(0,2)}(p, q) = B(p, q),$$

$$\begin{aligned} \tilde{W}^{(g,h+1)}(p, p_1, \dots, p_h) &= \sum_{q_i} \text{Res}_{q=q_i} \frac{dE_{q,\bar{q}}(p)}{\omega(q) - \omega(\bar{q})} \left(\tilde{W}^{(g-1,h+2)}(q, \bar{q}, p_1, \dots, p_h) \right. \\ &\quad \left. + \sum_{l=0}^g \sum_{J \subset H} \tilde{W}^{(g-l, |J|+1)}(q, p_J) \tilde{W}^{(l, |H|-|J|+1)}(\bar{q}, p_{H \setminus J}) \right), \end{aligned}$$

$$H = \{1, \dots, h\}, J = \{i_1, \dots, i_j\} \subset H, p_J = \{p_{i_1}, \dots, p_{i_j}\},$$

$$dE_{q,\bar{q}}(p) = \frac{1}{2} \int_q^{\bar{q}} B(p, \psi), \quad \text{near a ramification point } q_i.$$

Moreover, in [2], they defined $F^{(g)}$ ($g \in \mathbb{Z}, g \geq 2$) on Σ by

$$F^{(g)} = \frac{(-1)^g}{2-2g} \sum_{q_i} \text{Res}_{q=q_i} \theta(q) W^{(g,1)}(q),$$

where $\theta(q)$ is any primitive of $\omega(q)$ given by $d\theta(q) = \omega(q)$. And $F^{(1)}$ is defined separately as

$$F^{(1)} = -\frac{1}{2} \log \tau_B - \frac{1}{24} \log \prod_i \omega'(q_i),$$

where $\omega'(q_i) = \frac{1}{dz_i(p)} d\left(\frac{\log y(x)}{x}\right) \Big|_{p=q_i}$, $z_i(p) = \sqrt{x(p) - x(q_i)}$ and τ_B is the Bergmann tau-function [7].

Then the BKMP conjecture for toric Calabi-Yau 3-fold can be formulated as follow (Conjecture 1 in [4]).

Conjecture 2.1. *Let Σ_f be the framed mirror curve to a toric Calabi-Yau threefold X .*

1. The free energies $F^{(g)}$ constructed by the Eynard-Orantin recursion are mapped by the mirror map to the genus g generating functions of Gromov-Witten invariants of X .

2. The correlation functions $F^{(g,h)}$ are mapped by the open/closed mirror map to the generating functions of framed open Gromov-Witten invariants.

3. BKMP CONJECTURE FOR THE CASE OF \mathbb{C}^3

In this section, we restrict us to consider the special toric Calabi-Yau 3-fold \mathbb{C}^3 which has the framed mirror curve

$$\Sigma_f = \{H(x, y) := x + y^f + y^{f+1} = 0\} \subset (\mathbb{C}^*)^2.$$

Σ_f has only one ramification point

$$y_* = \frac{-f}{(f+1)}, \quad x_* = \frac{f^f}{(-1-f)^{-1-f}}.$$

By the definition of $\theta(q)$ given in Section 2, we have

$$(1) \quad \theta(y) = \frac{f}{2}(\log y)^2 + \log y \log(1+y) + Li_2(-y).$$

We define the differential form $\Psi_n(y; f)$ for $n \geq 0$ as follow:

$$(2) \quad \Psi_n(y; f) = -dy \frac{((1+f)y+f)}{y(y+1)} \left(\frac{y(y+1)}{(1+f)y+f} \frac{d}{dy} \right)^{n+1} \frac{1}{(1+f)((1+f)y+f)}.$$

For examples, when $n = 0$ and 1:

$$\begin{aligned} \Psi_0(y; f) &= dy \frac{1}{(f+(f+1)y)^2}; \\ \Psi_1(y; f) &= -dy \frac{3(1+f)y(y+1) - (1+2y)(f+(f+1)y)}{(f+(f+1)y)^4}. \end{aligned}$$

For convenience, in the following exposition, we also introduce the notation $\hat{\Psi}_n(y; f)$ by the relationship $\Psi_n(y; f) = -\hat{\Psi}_n(y; f)dy$.

By the Eynard-Orantin topological recursions introduced in Section 2, we have

$$\begin{aligned} W^{(0,3)}(y_1, y_2, y_3) &= -(f(f+1))^2 \Psi_0(y_1; f) \Psi_0(y_2; f) \Psi_0(y_3; f); \\ W^{(0,4)}(y_1, y_2, y_3, y_4) &= (f(f+1))^3 \sum_{i=1}^4 \Psi_1(y_i; f) \prod_{j \neq i} \Psi_0(y_j; f); \\ W^{(1,1)}(y) &= \frac{1}{24} ((1+f+f^2)\Psi_0(y; f) - f(f+1)\Psi_1(y; f)); \\ W^{(1,2)}(y_1, y_2) &= \frac{1}{24} (-(1+f+f^2)\Psi_0(y_1; f)\Psi_1(y_2; f) \\ &\quad + f(1+f)\Psi_0(y_1; f)\Psi_2(y_2; f) + (y_1 \leftrightarrow y_2) + f(1+f)\Psi_1(y_1; f)\Psi_1(y_2; f)); \\ W^{(2,1)}(y) &= \frac{1}{5760} (2f(f+1)\Psi_1(y; f) - 7(1+f+f^2)^2\Psi_2(y; f) \\ &\quad + 12f(1+2f+2f^2+f^3)\Psi_3(y; f) - 5f^2(f+1)^2\Psi_4(y; f)). \end{aligned}$$

For the general $g \geq 2$ and $h \geq 1$, L. Chen [6] and J. Zhou [15] have proved the following identity independently (See also [16]).

$$W^{(g,h)}(y_1, \dots, y_h) = (-1)^{g+h} (f(f+1))^{h-1} \sum_{n_i \geq 0} \left\langle \prod_{i=1}^h \tau_{n_i} \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \right\rangle_g \prod_{i=1}^h \Psi_{n_i}(y_i; f)$$

where $\Lambda_g^\vee(t) = t^g - t^{g-1}\lambda_1 + \dots + (-1)^g \lambda_g$.

In particularly,

$$W^{(g,1)}(y) = (-1)^{g+1} \sum_{n \geq 0}^{2g-2} \langle \tau_n \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g \Psi_b(y; f).$$

Thus the free energy part of the BKMP conjecture for the case of \mathbb{C}^3 is given by

$$\begin{aligned} F^{(g)} &= \frac{(-1)^g}{2-2g} \text{Res}_{y=\frac{-f}{1+f}} \theta(y) W^{(g,1)}(y) \\ &= \frac{1}{2g-2} \sum_{n \geq 0} \langle \tau_n \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g \text{Res}_{y=\frac{-y}{1+y}} \theta(y) \Psi_n(y; f). \end{aligned}$$

Then Conjecture 1.1 will be finished by the following two lemmas and the Hodge integral identity [11, 13],

$$\langle \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}.$$

Lemma 3.1. *The degree $3g-3$ part of $\Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f)$ is given by*

$$(-1)^{g-1} f(f+1) \lambda_g \lambda_{g-1} \lambda_{g-2}.$$

Proof. By Mumford's relation: $\Lambda_g^\vee(1) \Lambda_g^\vee(-1) = (-1)^g$, we have $\lambda_{g-1}^2 = 2\lambda_g \lambda_{g-2}$, $\lambda_g^2 = 0$. Then the degree $3g-3$ part of $\Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f)$ is equal to

$$\begin{aligned} &((-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} + (-1)^{g-2} \lambda_{g-2}) \times (-1)^g (\lambda_g + (f+1) \lambda_{g-1} + (f+1)^2 \lambda_{g-2}) \\ &\times ((-1)^g \lambda_g + f(-1)^{g-1} \lambda_{g-1} + f^2(-1)^{g-2} \lambda_{g-2}) \\ &= (-1)^{g-1} f(f+1) \lambda_g \lambda_{g-1} \lambda_{g-2}. \end{aligned}$$

□

Lemma 3.2.

$$\text{Res}_{y=\frac{-f}{1+f}} \theta(y) \Psi_n(y; f) = \begin{cases} -\frac{1}{f(1+f)}, & n=1, \\ 0, & n \geq 2 \text{ or } n=0. \end{cases}$$

Proof. Let $z = y + \frac{f}{1+f}$, by formula (1)

$$\theta(z) = \frac{f}{2} \left(\log \left(z - \frac{f}{1+f} \right) \right)^2 + \log \left(z - \frac{f}{1+f} \right) \log \left(z + \frac{1}{1+f} \right) + Li_2 \left(-z + \frac{f}{1+f} \right).$$

Hence,

$$d\theta(z) = \frac{(1+f)z \log \left(z - \frac{f}{1+f} \right)}{\left(z - \frac{f}{1+f} \right) \left(z + \frac{1}{1+f} \right)} dz.$$

From formula (2),

$$(4) \quad \hat{\Psi}_n(z; f) = -\frac{d}{dz} \left(\hat{\Psi}_{n-1}(z; f) \frac{\left(z - \frac{f}{1+f} \right) \left(z + \frac{1}{1+f} \right)}{(1+f)z} \right) \text{ for } n \geq 1.$$

and

$$\hat{\Psi}_0(z; f) = -\frac{1}{(1+f)^2 z}.$$

By the recursion formula (4), it is easy to show that $\hat{\Psi}_n(z; f)$ has the following form

$$\hat{\Psi}_n(z; f) = \frac{a_0(f) + a_1(f)z + \cdots + a_{2n}(f)z^{2n}}{((1+f)z)^{2n+2}}$$

where $a_0(f), \dots, a_{2n}(f)$ are some polynomials of framing f .

From the residue identity,

$$0 = \text{Res}_{z=0} d(f(z)g(z)) = \text{Res}_{z=0} g(z)df(z) + \text{Res}_{z=0} f(z)dg(z)$$

We have

$$(5) \quad \text{Res}_{z=0} g(z)df(z) = -\text{Res}_{z=0} f(z)dg(z).$$

Formula (5) will be used iteratively in the following exposition.

When $n = 0$,

$$\begin{aligned} & \text{Res}_{y=\frac{-f}{1+f}} \theta(y) \Psi_0(y; f) \\ &= -\text{Res}_{y=-\frac{f}{1+f}} \theta(y) \hat{\Psi}_0(y; f) dy \\ &= -\text{Res}_{z=0} \theta(z) \hat{\Psi}_0(z; f) dz \\ &= -\text{Res}_{z=0} \theta(z) \frac{1}{(1+f)^2} d\left(\frac{1}{z}\right) \\ &= \text{Res}_{z=0} \frac{1}{(1+f)^2 z} d\theta(z) \\ &= \text{Res}_{z=0} \frac{\log\left(z - \frac{f}{1+f}\right)}{(1+f) \left(z - \frac{f}{1+f}\right) \left(z + \frac{1}{1+f}\right)} \\ &= 0. \end{aligned}$$

When $n = 1$,

$$\begin{aligned} & \text{Res}_{y=\frac{-f}{1+f}} \theta(y) \Psi_1(y; f) \\ &= -\text{Res}_{y=-\frac{f}{1+f}} \theta(y) \hat{\Psi}_1(y; f) dy \\ &= -\text{Res}_{z=0} \theta(z) \hat{\Psi}_1(z; f) dz \\ &= \text{Res}_{z=0} \theta(z) d\left(\hat{\Psi}_0(z; f) \frac{\left(z - \frac{f}{1+f}\right) \left(z + \frac{1}{1+f}\right)}{(1+f)z}\right) \\ &= -\text{Res}_{z=0} \hat{\Psi}_0(z; f) \log\left(z - \frac{f}{1+f}\right) dz \\ &= \text{Res}_{z=0} \frac{1}{(1+f)^2 z^2} \log\left(z - \frac{f}{1+f}\right) dz \\ &= -\frac{1}{f(1+f)} \end{aligned}$$

More generally, when $n \geq 2$

$$\begin{aligned} & \text{Res}_{y=\frac{-f}{1+f}} \theta(y) \Psi_n(y; f) \\ &= -\text{Res}_{y=-\frac{f}{1+f}} \theta(y) \hat{\Psi}_n(y; f) dy \\ &= -\text{Res}_{z=0} \theta(z) \hat{\Psi}_n(z; f) dz \end{aligned}$$

$$\begin{aligned}
&= \text{Res}_{z=0} \theta(z) d \left(\hat{\Psi}_{n-1}(z; f) \frac{\left(z - \frac{f}{1+f}\right) \left(z + \frac{1}{1+f}\right)}{(1+f)z} \right) \\
&= -\text{Res}_{z=0} \hat{\Psi}_{n-1}(z; f) \frac{\left(z - \frac{f}{1+f}\right) \left(z + \frac{1}{1+f}\right)}{(1+f)z} d\theta(z) \\
&= -\text{Res}_{z=0} \hat{\Psi}_{n-1}(z; f) \log \left(z - \frac{f}{1+f} \right) dz \\
&= \text{Res}_{z=0} d \left(\hat{\Psi}_{n-2}(z; f) \frac{\left(z - \frac{f}{1+f}\right) \left(z + \frac{1}{1+f}\right)}{(1+f)z} \right) \log \left(z - \frac{f}{1+f} \right) \\
&= -\text{Res}_{z=0} \hat{\Psi}_{n-2}(z; f) \frac{\left(z + \frac{1}{1+f}\right)}{(1+f)z} \\
&= -\text{Res}_{z=0} \frac{a_0(f) + a_1(f)z + \cdots + a_{2n-4}(f)z^{2n-4}}{((1+f)z)^{2n-2}} \left(\frac{1}{(1+f)} + \frac{1}{(1+f)^2 z} \right) \\
&= 0.
\end{aligned}$$

□

Now, we can finish the proof of Conjecture 1.1 by lemma 3.1 and 3.2.

Proof.

$$\begin{aligned}
F^{(g)} &= \frac{(-1)^g}{2-2g} \text{Res}_{y=-\frac{f}{1+f}} \theta(y) W^{(g,1)}(y) \\
&= \frac{1}{2g-2} \sum_{n=1}^{3g-2} \langle \tau_n \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g \text{Res}_{y=-\frac{f}{1+f}} \theta(y) \Psi_n(y; f) \\
&= -\frac{1}{2g-2} \frac{1}{f(1+f)} \langle \tau_1 \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g \\
&= \frac{(-1)^g}{2g-2} \langle \tau_1 \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g \\
&= (-1)^g \langle \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g \\
&= (-1)^g \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}
\end{aligned}$$

where we have used the dilaton equation for Hodge integrals

$$\langle \tau_1 \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g = (2g-2) \langle \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g.$$

Thus the Conjecture 1.1 is proved. □

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